SECTION 12.3: THE CALCULUS OF POLAR EQUATIONS

SLOPES OF POLAR CURVES:

We can view the graph of a polar function $r = f(\theta)$ as the graph of the parametric equations:

$$\{x = r\cos(\theta) = f(\theta)\cos(\theta), \qquad y = r\sin(\theta) = f(\theta)\sin(\theta)$$

As a result, we can use what we learned about the Calculus of parametric equations to get

$$\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

TANGENTS AT THE POLES:

If $r(\alpha) = 0$ but $r'(\alpha) \neq 0$, then $\theta = \alpha$ is a tangent line to the polar curve $r = f(\theta)$ at the pole $r = 0$.

PROOF: Since $r(\alpha) = 0$, we know $r = f(\theta)$ reaches the pole when $\theta = \alpha$. Since $r'(\alpha) \neq 0$, we get:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,0)} = \frac{r'(\alpha)\sin(\alpha) + r(\alpha)\cos(\alpha)}{r'(\alpha)\cos(\alpha) - r(\alpha)\sin(\alpha)} = \frac{r'(\alpha)\sin(\alpha)}{r'(\alpha)\cos(\alpha)} = \tan(\alpha)$$

Hence, the tangent line through the pole, $(x, y) = (0, 0)$ is $y = \tan(\alpha)\,x$ which, in polar coordinates, is $\theta = \alpha$.

EXAMPLE 1: Consider the polar curve $r = f(\theta) = 1 - 2\sin(\theta)$.

1. Write the equation of the tangent line at $(x, y) = (-1, 0)$.

   Ans: First we note that the complete graph of $r = 1 - 2\sin(\theta)$ is traced out over the interval $0 \leq \theta < 2\pi$.

   Next we find a value of θ which produces the point $(x, y) = (-1, 0)$ on the graph of $r = 1 - 2\sin(\theta)$.

   In this instance, we come across $\theta = \pi$ as one such value in the course of graphing.

   If we were asked to find the tangent line at more exotic location, we'd have to solve a system of equations:

   $$x = r\cos(\theta) = (1 - 2\sin(\theta))\cos(\theta) = 1, \quad \text{and} \quad y = (1 - 2\sin(\theta))\sin(\theta) = 0.$$

   Next, we find:

   $$\frac{dx}{d\theta} = D_\theta\left[f(\theta)\cos(\theta)\right] = D_\theta\left[(1 - 2\sin(\theta))\cos(\theta)\right] = ... = -2\cos(2\theta) - \sin(\theta) \text{ so that } \left.\frac{dx}{d\theta}\right|_{\theta=\pi} = -2$$

   $$\frac{dy}{d\theta} = D_\theta\left[f(\theta)\sin(\theta)\right] = D_\theta\left[(1 - 2\sin(\theta))\sin(\theta)\right] = ... = \cos(\theta)(1 - 4\sin(\theta)) \text{ so that } \left.\frac{dy}{d\theta}\right|_{\theta=\pi} = -1$$

   Hence, $\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{-1}{-2} = \frac{1}{2}$. This gives the tangent line: $y = \frac{1}{2}(x + 1) + 0$ or $y = \frac{1}{2}x + \frac{1}{2}$.

2. Locate the points where the curve $r = 1 - 2\sin(\theta)$ has a horizontal tangent line.

   Ans: To find horizontal tangents, we set the numerator of $\dfrac{dy}{dx}$, namely $\dfrac{dy}{d\theta} = 0$:

$$f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) = ... = \cos(\theta)(1 - 4\sin(\theta)) = 0$$

   From $\cos(\theta) = 0$, we get $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

   From $1 - 4\sin(\theta) = 0$, we get $\sin(\theta) = \frac{1}{4}$. Hence, $\theta = \sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right)$.

   Since none of these values θ make the denominator of $\dfrac{dy}{dx} = 0$, we substitute these values of θ into $r = f(\theta)$:

   - $r\left(\frac{\pi}{2}\right) = -1$ which corresponds to $(x, y) = (0, -1)$
   - $r\left(\frac{3\pi}{2}\right) = 3$ which corresponds to $(x, y) = (0, -3)$
   - $r\left(\sin^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{2}$ which corresponds to $(x, y) = \left(\frac{\sqrt{15}}{8}, \frac{1}{8}\right)$
   - $r\left(\pi - \sin^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{2}$ which corresponds to $(x, y) = \left(-\frac{\sqrt{15}}{8}, \frac{1}{8}\right)$

   Desmos confirms our solutions.

AREA IN POLAR COORDINATES:

In the same way we used rectangles to approximate areas in rectangular coordinates, we can use central sectors of circles to approximate areas in polar coordinates. Doing so gives us the following:

The **area between the pole and the curve** $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is given by: $\dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$

STRATEGY FOR FINDING AREAS IN POLAR COORDINATES:

Sketch a detailed graph of the region and exploit symmetry whenever you can.

EXAMPLE 2: Find the area of the following:

1. The area enclosed by the inner loop of the graph of $r = 1 - 2\sin(\theta)$.

   Ans: $A = \dfrac{1}{2}\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [1 - 2\sin(\theta)]^2 \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [1 - 2\sin(\theta)]^2 \, d\theta = \ldots = \pi - \dfrac{3\sqrt{3}}{2}$ units^2

2. The area between the loops of the graph of $r = 1 - 2\sin(\theta)$.

   Ans: $A = \dfrac{1}{2}\displaystyle\int_{0}^{\frac{\pi}{6}} [1 - 2\sin(\theta)]^2 \, d\theta + \dfrac{1}{2}\int_{\frac{5\pi}{6}}^{2\pi} [1 - 2\sin(\theta)]^2 \, d\theta - \left(\pi - \dfrac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$ units^2

   or, using symmetry:

   $A = \displaystyle\int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} [1 - 2\sin(\theta)]^2 \, d\theta - \left(\pi - \dfrac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$ units^2

3. The area enclosed by one leaf of $r = 4\cos(2\theta)$.

   Ans: $A = \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [4\cos(2\theta)]^2 \, d\theta = \int_{0}^{\frac{\pi}{4}} [4\cos(2\theta)]^2 \, d\theta = 2\pi$ units^2

4. The 'common interior' of $r = 4\cos(2\theta)$ and $r = 2$:

   Ans: $A = 8\left[\dfrac{1}{2}\displaystyle\int_{0}^{\frac{\pi}{6}} (2)^2 \, d\theta + \dfrac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} [4\cos(2\theta)]^2 \, d\theta\right] = \dfrac{16\pi}{3} - 4\sqrt{3}$ units^2

HOMEWORK: Section 12.3: 9 - 61 every other odd